ACYCLIC COLORINGS OF PLANAR GRAPHS†

BY

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ABSTRACT

A coloring of the vertices of a graph by k colors is called acyclic provided that no circuit is bichromatic. We prove that every planar graph has an acyclic coloring with nine colors, and conjecture that five colors are sufficient. Other results on related types of colorings are also obtained; some of them generalize known facts about "point-arboricity".

1. Introduction

Let G denote a graph with vertex set V; we shall assume th a G contains no 1- or 2-circuits (that is, loops or multiple edges). A *k-coloring* of G is a partition $V = V_1 \cup \cdots \cup V_k$ of the vertices of G into k pairwise disjoint sets (called *colors*) so that adjacent vertices are in different sets (have different colors). A k-coloring of G is called *acyclic* provided that every subgraph of G spanned by vertices of two of the colors is acyclic (in other words, is a forest). If G is the graph of the octahedron then the 4-coloring of G indicated in Fig. 1 by the numerals placed near the vertices is not acyclic (since the colors 1 and 2 span a graph which is not

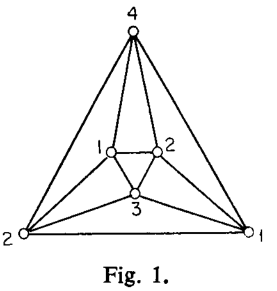

Fig. 1.

† Research supported in part by the Office of Naval Research under Grant N00014–67–A–0103–0003.

Received August 28, 1972

acyclic); but the 5-coloring of G indicated in Fig. 2 is acyclic. More generally, if G is the graph of any bipyramid with four or more sides then five colors are necessary for an acyclic coloring of G.

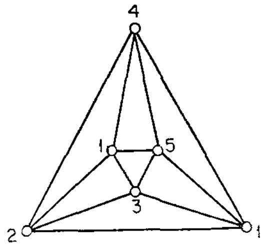

Fig. 2.

Our main justification for introducing the notion of acyclic colorings is the following

CONJECTURE 1. *Every planar graph has an acyclic 5-coloring.*

The conjecture seems to merit some attention on two counts:

First, it appears to be rather hard. As a matter of fact, if one fixes any value of k there seems to be no obvious proof that every planar graph is acyclically k-colorable. By a somewhat involved argument we shall prove (in Theorem 1) that every planar graph has acyclic 9-colorings.

Second, the truth of the conjecture would imply certain results on "point-arboricity" due to Chartrand, Kronk and Wall [4], Hedetniemi [12], Chartrand and Kronk [3], Stein [23] and Chartrand, Geller and Hedetniemi [1]. Although those results do not follow from our Theorem 1, we shall establish (in Theorem 2) a weakened version of the conjecture which is sufficient to imply stronger and more symmetric variants of those results on "point-arboricity".

The main result of the present paper is formulated and proved in Section 2. Section 3 contains an exposition of the results on "point-arboricity" as well as our Theorem 2 and its proof. Various remarks, problems and conjectures are collected in Section 4.

The author is indebted to S. Hedetniemi, G. Kalai and G. Wegner for helpful suggestions.

2. The main result

In this section we shall prove:

THEOREM 1. *Every planar graph G is acyclically 9-colorable.*

In the proof of this theorem we clearly may assume, without loss of generality, that G has at least six vertices and that G is a maximal planar graph, that is, a possibly curvilinear triangulation of the plane (or the 2-sphere). With these assumptions, the steps of the proof may be described as follows:

(1) After choosing an arbitrary starting vertex v_0 of G, we associate with each other vertex v_i its *distance* $\mathrm{d}(v_i)$ from v_0, which is the least number of edges in a path in G connecting v_0 and v_i. We define a partition of the vertices of G by putting $V_0 = \{v_0\}$ and $V_j = \{v_i \in V \mid \mathrm{d}(v_i) = j\}$ for $j = 1, 2, \cdots, n$, where n is the largest of the numbers $\mathrm{d}(v_i)$.

(2) For each j, $0 \leqq j \leqq n$, we shall form a graph G_j having vertex set V_j and determined by a well-defined construction using G. Each G_j will turn out to belong to a special type of planar graphs which we call diagonalized polygons. (This felicitous term was suggested by G. Wegner.)

(3) Each diagonalized polygon is acyclically 3-colorable.

(4) We color the vertices of each V_j by three colors by taking acyclic 3-colorings of $G_0, G_3, \cdots, G_i$ $(i \equiv 0 \bmod 3), \cdots$ with one set of three colors, acyclic 3-colorings of $G_1, G_4, \cdots, G_i$ $(i \equiv 1 \bmod 3), \cdots$ with another three colors, and acyclic 3-colorings of $G_2, G_5 \cdots, G_i$ $(i \equiv 2 \bmod 3)$ with a third set of three colors. Thus we obtain a 9-coloring of G which we then show to be acyclic.

We turn now to the details.

Step 1. The partition $V = V_0 \cup V_1 \cup \cdots \cup V_n$ of the vertex set V of G into disjoint sets V_j is clearly uniquely determined by the choice of the initial vertex v_0. It is also clear that if $v_i \in V_i$ and $v_j \in V_j$ are the endpoints of an edge of G then either $i = j + 1$ or $i = j$ or $i = j - 1$.

Step 2. G_0 is the graph consisting of the single vertex v_0 and having no edges. Before defining the other G_j, we take an imbedding of G in the 2-sphere S and, taking v_0 as the north pole of S, we find the stereographic projection G_1^* into a plane, tangent at the south pole of S, of the graph obtained from G by omiting v_0 (and the edges incident with v_0). In order to simplify the notation, we shall denote the vertices of G_1^* by the same symbols as the corresponding vertices of G. The vertices constituting V_1 are clearly just those vertices of G_1^* that are adjacent to the unbounded face of G_1^*. We now define inductively graphs G_j^* for $2 \leqq j \leqq n$. If we have already formed the graph G_{j-1}^* so that precisely the vertices of the set V_{j-1} are the vertices of G_{j-1}^* adjacent to the unbounded face of G_{j-1}^*, then we omit from G_{j-1}^* the vertices in V_{j-1} (and the edges incident with them), and call the remaining graph G_j. It is easily verified that V_j is precisely the set of those vertices of G_j^* that are adjacent to the unbounded face of G_j^*.

Next, let C_j be the subgraph of G_j^* spanned by the set of vertices V_j. Then it is seen that C_1 consists of a circuit with possibly some of its diagonals, but that C_j for $j > 1$ may happen to be only 1-connected or even disconnected. (The notation and constructions are illustrated in Fig. 3, where $n = 3$.) Clearly $G_n^* = C_n$. From the graph C_j we obtain a graph C_j^* as follows:

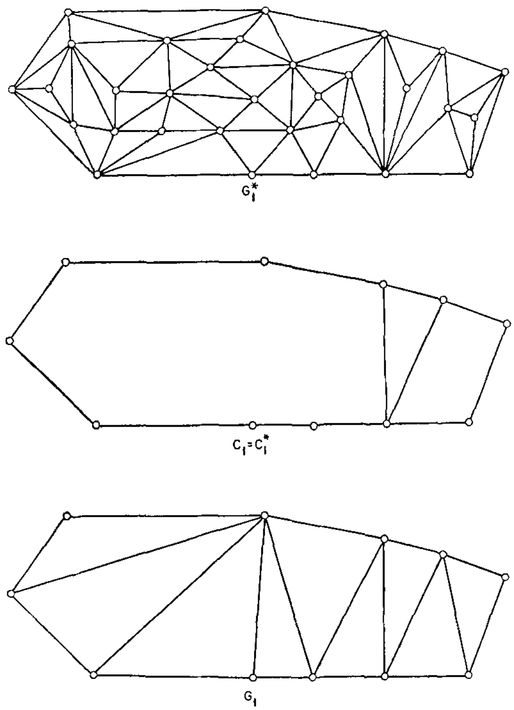

Fig. 3a.

For $j = n$ we have $C_n^* = C_n$.

For $1 \leqq j \leqq n-1$, C_j^* consists of C_j and, possibly, additional edges between vertices in the vertex set V_j of C_j. In order to see what edges, if any, should be added to C_j in forming C_j^* we consider, in turn, each of the vertices belonging to V_{j+1} in the graph G_j^*. With each vertex $w \in V_{j+1}$ we have these alternatives:

(i) w is adjacent to only one vertex in V_j; then w causes no edges to be added.

(ii) w is adjacent to precisely two vertices v_1 and v_2 of V_j; then we add to C_j the edge (v_1, v_2) unless it is already present in C_j.

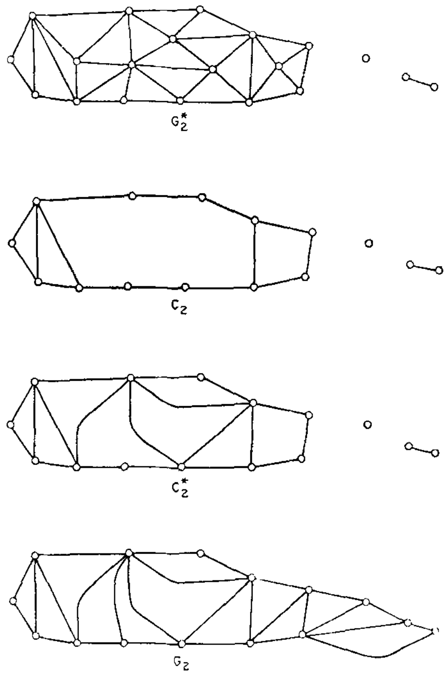

Fig. 3b.

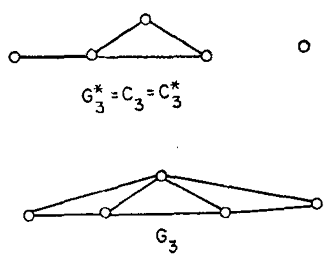

Fig. 3c.

(iii) w is adjacent to m vertices of V_j, where $m \geqq 3$. Let those vertices be v_1, $v_2, \cdots, v_m, v_{m+1} = v_1$, the notation indicating the clockwise order in which the edges (w,v_i) follow each other. Then in the formation of C_j^* all the edges (v_i,v_{i+1}), $i = 1,2,\cdots,m$, are added unless they are already present in C_j or have been added while considering some other vertex in V_{j+1}.

In cases (ii) and (iii) each added edge (v_i,v_{i+1}) is supposed to be imbedded in the plane suitably close to the arc composed of (v_1,w) and (w,v_{i+1}). Hence each C_j^* is planar, and all the vertices of C_j^* are adjacent to the unbounded face of C_j^*.

We form G_j from C_j^*, for $1 \leqq j \leqq n$, by adding, if necessary, edges between the vertices of C_j^* in such a manner and number that all the vertices in V_j, the vertex set of G_j, are adjacent to the unbounded face of G_j, and all the other faces of G_j are triangles. Thus each G_j is either a single vertex, or an edge, or a simple circuit partitioned by diagonals into triangles. In the sequel we shall call such graphs "diagonalized polygons". (Some authors call them "maximal outerplanar graphs".)

Step 3. We shall now establish, by induction on the number m of vertices o the polygon, that each diagonalized polygon P_m is acyclically 3-colorable. Clearly we may assume that $m \geqq 4$. As is well known (and easily established either by induction or by a direct argument) there exist in P_m at least two 2-valent vertices. Let v_m be a 2-valent vertex of P_m, and let v_1 and v_{m-1} be the two vertices of P_m adjacent to v_m. Then P_m contains the edge (v_1, v_{m-1}), and the graph P_{m-1} obtained from P_m by omitting v_m is a diagonalized $(m-1)$-gon. By the inductive assumption P_{m-1} may be acyclically 3-colored. Given any acyclic 3-coloring of P_{m-1} we obtain from it an acyclic 3-coloring of P_m by using the same colors for the vertices of P_m different from v_m, and by assigning to v_m the color different from the two colors assigned to v_1 and v_{m-1}.

Step 4. We have partitioned the vertex set V of G into subsets $V_0, V_1, \cdots, V_n$. Each V_j is also the vertex set of a diagonalized polygon G_j. We choose a fixed acyclic 3-coloring of each G_j, using the same set of three colors for each of the graphs G_0, G_3, $G_6, \cdots$, a set of three different colors for G_1, G_4 $G_7, \cdots$, and a set of still other three colors for G_2, G_5, G_8, $\cdots$etc. Thus each vertex of G is assigned one of the nine colors used. We shall show that this 9-coloring of G is acyclic. Assume, on the contrary, that α and β are two of the nine colors and that $v_0, v_1, \cdots, v_{2p} = v_0$ form a simple circuit K in which all vertices have one of the colors α or β. Then the set $W = \{v_0, v_1, \cdots, v_{2p}\}$ of vertices of K cannot be a subset of one of the graphs G_j, since the coloring of each G_j was acyclic. Thus W meets at least two of the sets V_j and since the circuit K is connected, the sets V_j met by W must have consecutive subscripts. But since only two colors α and β are available for the vertices in W, it follows that W meets precisely two sets V_j, say V_r and V_{r+1}. Assume, for definiteness, that v_0, $v_2, \cdots$ are in V_r and have color α while v_1, $v_3, \cdots$ are in V_{r+1} and have color β. We claim that this is also impossible by the construction of G_r. Indeed, consider the vertex $v_1 \in V_{r+1}$. Since v_0 and v_2 are both adjacent to v_1 but have in the acyclic coloring of G_r the same color α, it follows that in G_r^*

the edges (v_1, v_0) and (v_1, v_2) are cyclically separated by some edges (v_1, v^*) and (v_1, v^{**}), where v^* and v^{**} are in V_r and are chosen as close as possible to v_0 in the cyclic ordering of the edges issuing from v_1 and having their other vertex in V_r. Then v^* and v^{**} are adjacent to v_0 in G_r and therefore neither has color α; since they belong to V_r, neither has color β either. The contradiction we were seeking now results on observing that the vertices v^*, v_1, v^{**} constitute a cut in the graph G_r^* which makes it impossible for the path $v_2, v_3, \cdots, v_{2j} = v_0$ which is contained in G_r^* and misses v_1 to be colored only with colors α and β.

Hence the 9-coloring of G we obtained is acyclic, and the proof of Theorem 1 is completed.

3. Partially acyclic colorings

Let k be a positive integer, and let $k_1, \cdots, k_s$ be positive integers such that $k = k_1 + \cdots + k_s$. We shall say that a k-coloring of a graph G is a *partially acyclic k-coloring of type* $(k_1, \cdots, k_s)$ of G, or more succinctly, a $(k_1, \cdots, k_s)$*-coloring* of G, if the k colors may be partitioned into sets of $k_1, k_2, \cdots, k_s$ colors such that any pair of colors taken from the same set span an acyclic subgraph of G. Thus an acyclic k-coloring is a (k)-coloring, and a k-coloring in the usual sense could also be termed a $(1,1,\cdots,1)$-coloring.

It was proved in [2], [3], [4] and [12] that the vertices of each planar graph G may be partitioned into three sets such that the subgraph of G spanned by each of the sets is acyclic. (In their terminology this result reads: "The point-arboricity of each planar graph is at most 3".) Since each acyclic graph (i.e., each forest) is acyclically 2-colorable, and since each acyclically 2-colorable set is a forest, the result just mentioned may be formulated as:

PROPOSITION 1. *Every planar graph has a* $(2,2,2)$*-coloring.*

This result was strengthened by Stein [23]; he showed that one of the three forests into which each planar graph is decomposable by Proposition 1 may be assumed to consist of isolated vertices. Thus Stein's result may be rendered as:

PROPOSITION 2. *Every planar graph has a* $(2,2,1)$*-coloring.*

We shall strengthen this still further by proving:

THEOREM 2. *Every planar graph G has a* $(2,3)$*-coloring.*

PROOF. We shall again assume, without loss of generality, that G has at least six vertices and that it is a triangulation of the plane. We shall prove, by induction on the number of vertices, the stronger assertion:

(*) *Every triangulation G of the plane has a* (2, 3)-*coloring*; *moreover, the three colors to be assigned to the vertices of one arbitrary triangle may be prescribed in advance.*

In the sequel, the triangle with prescribed coloring will be called the *distinguished* triangle.

The proof of (*) will be carried out in several steps. We shall first consider the case in which G is not 4-connected; next we shall treat the case in which G has a 4-valent vertex not incident with the distinguished triangle. Then we shall consider the case in which G is 4-connected and all its vertices have valence at least five; we shall show how to carry out the proof in case G contains either a (5, 5)-*edge* (that is, an edge both endpoints of which have valence 5), or a (5, 6)-*edge* (that is, an edge one endpoint of which has valence 5, the other valence 6), provided that the edge in question has no vertex in the distinguished triangle and none that belongs to a *proper* 4-*cut* of G. (A proper k-cut of G is a circuit of G of length k such that both components of its complement contain vertices of G.) Finally we shall show that one of these cases must occur.

Step 1. If G is not 4-connected, then G contains a proper 3-cut C. Let G^* and G^{**} be the two components into which C separates G, and let us assume that G^{**} contains the distinguished triangle. By the inductive assumption, G^{**} has a (2,3)-coloring with the required properties; also by the inductive assumption, G^* has a (2,3)-coloring that assigns to the vertices of C the same colors they obtained in the (2,3)-coloring of G^{**}. It is now immediate that the 5-coloring of G determined by the (2,3)-colorings of G^* and G^{**} is indeed the required (2,3)-coloring of G.

Step 2. Let v be a 4-valent vertex of G that does not belong to the distinguished triangle. We construct a graph G^* by deleting v from G and introducing an edge between two of the diagonally opposite vertices that were adjacent to v (see Fig. 4). Since G may be assumed to be 4-connected, either of the two possible choices is permissible. The way of obtaining a (2,3)-coloring of G from any (2,3)-coloring of G^* is described in Table 1 for all cases except the first one. In Table 1 (as well as in the other tables) colors α and β form the set of two and γ, δ and ε the set of three colors. Each row of the table indicates a possible coloring; the

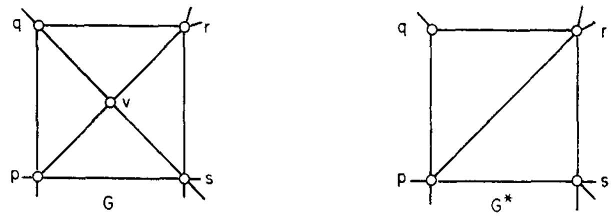

Fig. 4.

entries under p, q, r, s indicate all the possibilities (up to permutations) for G^*, the entries under v a suitable choice for G.

TABLE 1

p	q	r	s	v
α	γ	β	γ	see text
α	γ	β	δ	ε
α	β	γ	β	ε
α	β	γ	δ	ε
α	δ	γ	δ	β
α	δ	γ	ε	β
γ	α	δ	α	ε
γ	α	δ	β	ε
γ	α	δ	ε	β
γ	ε	δ	ε	β

In the case described in the first row of Table 1 we consider the maximal connected subgraph H of G^* containing only vertices colored α and β, and containing the vertices p and r. Clearly H is a tree that contains the edge pr. To obtain the required coloring of G we assign v the color β and interchange the colors α and β in that subtree of H that contains r and is obtained from H by deleting the edge pr.

Step 3. We now assume that G is 4-connected and that it contains an edge vw having one vertex of valence 5 and the other of valence 5 or 6, such that vw does not meet the distinguished triangle and such that (see Figs. 5 and 6) the vertices p and p' in the first case, and q, q' and r, r' in the second have no common neighbor outside the subgraph shown. By omitting the vertices v and w, and identifying p with p', respectively q with q' and r with r', we obtain from G a graph G^* which is (2,3)-colorable by the inductive hypothesis. To obtain a (2,3)-coloring of G from a given (2,3)-coloring of G^* we use Tables 2 and 3, and assigu to vertices with primed labels the colors of the vertices with corresponding

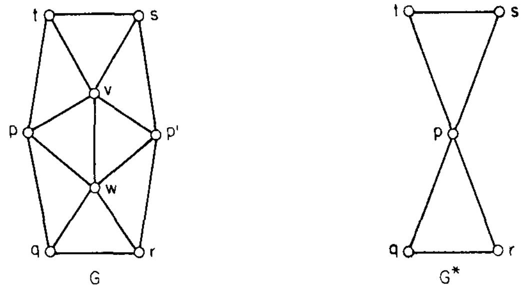

Fig. 5.

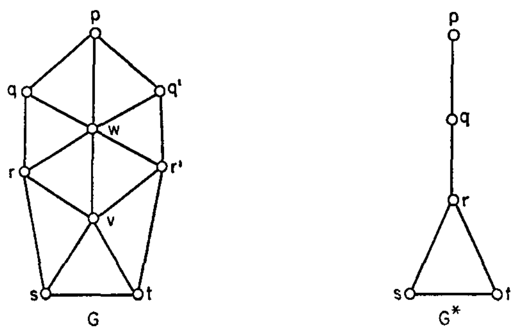

Fig. 6.

unprimed labels. In order to shorten the tables we have used the letters **a** and **b** to indicate either of α and β, and **c**, **d**, **e** to indicate arbitrary colors from among $\gamma, \delta, \varepsilon$, different latin characters designating different colors, and we used the symbol $\#$ to indicate an arbitrary color. All the choices of colors are understood to be made in accordance with the rule that adjacent vertices have different colors. Then for each possible (up to permutation) coloring of the graphs G^* of Figs. 5 and 6 we indicate in Tables 2 and 3 a suitable coloring of G.

TABLE 2.

p	q	r	s	t	v	w
γ	δ	ε	**c**	**d**	α	β
γ	δ	ε	α	$\#$	**c**	α
γ	α	δ	**c**	**a**	**b**	ε
γ	α	$\#$	α	**c**	**b**	**d**
γ	α	β	α	**b**	δ	ε
α	β	γ	β	**c**	**d**	**e**
α	β	γ	**c**	β	**d**	**e**
α	β	γ	**c**	**d**	β	δ
α	γ	δ	**c**	**d**	β	ε

TABLE 3.

p	q	r	s	t	v	w
#	#	α	β	γ	ε	δ
#	#	α	γ	δ	β	**c**
c	**d**	α	β	γ	δ	β
α	β	α	#	γ	ε	γ
α	β	γ	α	β	δ	ε
α	γ	δ	α	β	γ	β
γ	#	**c**	α	β	**d**	β
a	γ	δ	α	**c**	β	ε
c	α	γ	**a**	δ	ε	β
γ	δ	**c**	α	**d**	β	α
c	#	γ	δ	ε	**a**	**b**
c	**d**	α	γ	δ	ε	β

Step 4. In this, the last part of the proof of assertion (*), we have to show that one of the cases discussed in the previous steps takes place for every graph G. We first establish the following general result, in which $e_{j,k}$ denotes the number of edges of G having one vertex of valence j and the other of valence k.
(**) *If G is a triangulation of the plane and if each vertex of G has valence at least 5, then either $e_{5,5} > 0$ or else $e_{5,6} \geqq 60$.*

In order to prove (**), let v_k denote the number of k-valent vertices of G. If $e_{5,5} = 0$, consider the $5v_5$ edges incident with vertices of valence 5. Counting from the other endpoints of those edges and observing that two adjacent neighbors of a vertex of valence 6 or more may not both have valence 5, we have

$$5v_5 \leqq e_{5,6} + 3v_7 + 4v_8 + 4v_9 + 5v_{10} + \cdots.$$

On the other hand, from Euler's relation we have

$$v_5 = 12 + \sum_{k\geqq 7} (k-6)\, v_k,$$

so that

$$60 + \sum_{k\geqq 7} (5k - 30 - [k/2])\, v_k \leqq e_{5,6},$$

and in view of the non-negativity of the sum, $60 \leqq e_{5,6}$, as claimed.

Returning to the proof of assertion(*), in view of Step 1 we may assume that G is a 4-connected triangulation of the plane. If G contains a 4-valent vertex that is not incident with the distinguished triangle, we have the situation considered in Step 2. If all 4-valent vertices of G are incident with the distinguished triangle, there is a proper 4-cut Q that encloses the distinguished triangle. Among all such 4-cuts let Q_0 enclose the maximal possible number of faces of G. Then from each vertex of Q_0, at least two edges lead into the "outer" component of G. Therefore, if H denotes the graph obtained from G by deleting all the vertices that are inside Q_0 and replacing the inside of Q_0 by a copy of the part of G that is outside Q_0, then H is a triangulation with minimal valence 5, and all the vertices of Q_0 have valence at least 6 in H. By (**) there are in H either some (5,5)-edges or some (5,6)-edges; in either case there is in G an edge of the type required for Step 3.

We still have to deal with the case in which G is 4-connected but has no vertex of valence 4. If G has no proper 4-cuts, consider the graph H^* obtained by taking two copies of G and identifying the boundaries of the two distinguished triangles. Then the identified vertices have valences at least 8, and thus (5,5)-edges and (5,6)-edges of H^* correspond to edges of the same type in G that miss the distinguished triangle; thus we again may apply Step 3.

Finally, if G has a proper 4-cut Q we may assume that the distinguished triangle is inside Q; then we proceed, as above, to a maximal proper 4-cut Q_0, and are therefore again led to a situation in which Step 3 may be performed.

Thus one of the Steps 1, 2, 3 applies in each case and the proof of assertion (*), and of Theorem 2, is completed.

4. Remarks

(1) The concepts of acyclic and partially acyclic colorings lead to many open problems besides Conjecture 1. We already mentioned that forests coincide with the acyclically 2-colorable graphs. Clearly, each 2-coloring of a forest is also an acyclic coloring. Analogously, if P is a diagonalized polygon then, as we have seen in the proof of Theorem 1, P is acyclically 3-colorable; however, the same argument actually shows that every 3-coloring of P is an acyclic 3-coloring. More generally, if G is a graph that may be imbedded in the plane in such a manner that all its vertices are adjacent to the unbounded face, then G may be seen to be acyclically 3-colorable (but not every 3-coloring of such a graph needs to be acyclic); this is an obvious strengthening of "the (3, 2)-theorem" of [2]: "The point-arboricity of an outerplanar graph does not exceed 2". It would be of some interest to characterize those graphs for which every 3-coloring is acyclic.

As observed by G. Wegner (private communication), if G is a 3-colorable planar graph in which every simple circuit of even length has a diagonal, then every 3-coloring of G is acyclic. However, the unique 3-coloring of the graph G in Fig. 7 is acyclic although G contains a quadrangle without diagonals. This example (from a private communication by G. Wegner) refutes a conjecture made in earlier versions of this paper concerning the characterization of graphs having only acyclic 3-colorings.

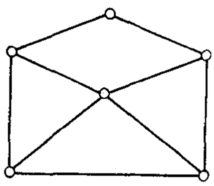

Fig. 7.

(2) There exist planar graphs G that are not (2,2)-colorable. Indeed, as observed by Stein [23] (correcting an erroneous assertion in [3]), if G is a triangulation of the plane then G is not (2,2)-colorable if and only if its dual G^* has no Hamiltonian circuit. The smallest known such graphs have 21 vertices (see Fig. 8 for one of them); their duals were independently discovered by D. W. Barnette (see [9]), Bosák [1] and Lederberg [17] (see [11] for more details concerning those graphs). The fact that the dual G^* of a 3-connected planar graph G has no Hamiltonian circuit does not imply that G is not (2,2)-colorable. For example, the graph in

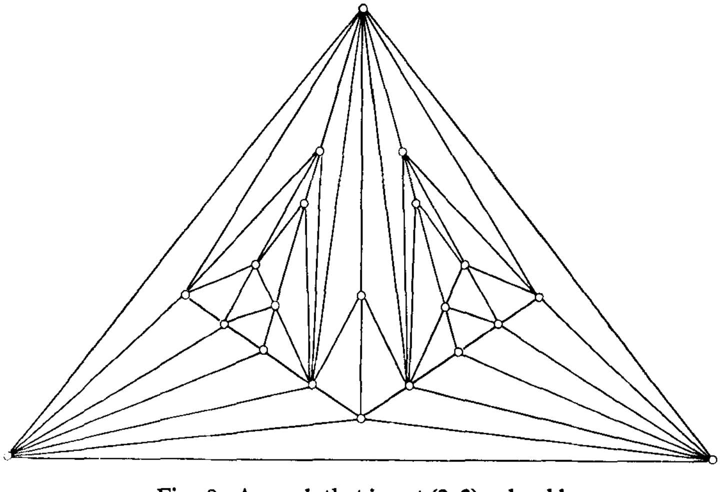

Fig. 8. A graph that is not (2, 2)-colorable.

Fig. 9 is (2.2)-colorable (α and β indicate one set of two colors, γ and ε the other) although its dual is easily seen not to have a Hamiltonian circuit.

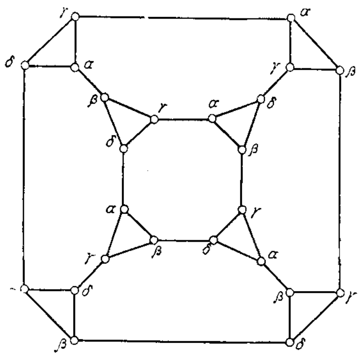

Fig. 9. A (2, 2)-colorable graph the dual of which has no Hamiltonian circuit.

(3) There exist planar graphs that are not (1,3)-colorable. The graph in Fig. 10 (with 14 vertices) may be verified to have that property; it is the smallest known planar graph which is not (1,3)-colorable.

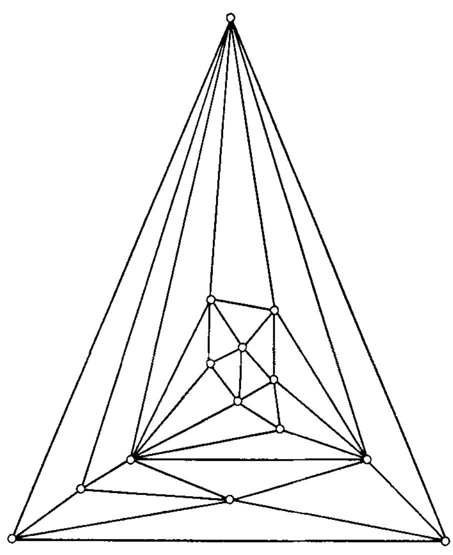

Fig. 10. A graph with no (1, 3)-coloring.

(4) The implication-diagrams in Fig. 11 indicate some of the open problems concerning acyclic and partially acyclic colorings. The type of coloring near the

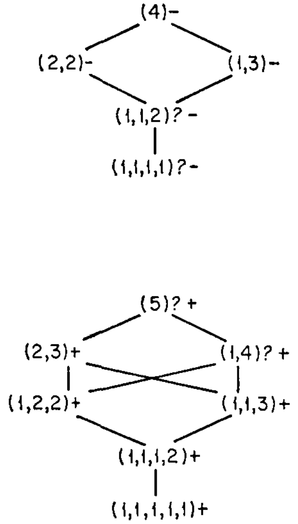

Fig. 11.

higher end of a slanted segment indicates a stronger assertion than the one near the lower end. The symbol + (or $-$) indicates that every (not every) planar graph has a coloring of that type. The symbols ?+ and ?$-$ indicate open problems, and our conjecture to their solution.

(5) Another open problem is the characterization of those planar graphs which have a (1,3)-coloring, of those that are (2,2)-colorable, and of those that may be acyclically 4-colored.

(6) By a slight refinement of the argument used in the proof of assertion (**) in Section 3 it may be shown that for each triangulation of the plane with all vertices of valence at least 5 we have $2e_{5,5}+e_{5,6} \geqq 60$. Similarly, one can show that for each triangulation of the plane either $e_{j,k}>0$ for some j and k such that $j+k \leqq 12$, or else $e_{3,10} \geqq 12$. The method we used in Section 3 to prove (**), which may also be made to yield the facts just mentioned, is due to Kotzig [13]. Kotzig used it to establish that every 3-connected planar graph G satisfies $e_{j,k}>0$ for some j and k with $j+k \leqq 13$, and that this happens even for some j, k with $j+k \leqq 11$ provided that all vertices of G have valence at least 5. It is rather curious that although the relation $e_{5,5}+e_{5,6}>0$ for triangulations with minimal valence 5 was established already in 1904 by Wernicke [24] and strengthened by Franklin [6] in 1922, Lebesgue [16] in 1940 and others, assertion (**) and Kotzig's results appear in none of those papers. It is also somewhat remarkable

that although assertion (**) may easily be established by the method of "Euler contributions" (see, for example, [21, Section 4.3]), Kotzig's result does not seem to be obtainable in that way.

(7) The definitions of acyclic k-coloring and partially acyclic colorings are meaningful also for graphs which are not necessarily planar. As an example of results possible in this direction we mention:

If G is a graph with maximal valence 3 *then G has an acyclic* 4-*coloring.*

The graph of the cube is an example of a 3-valent, 3-connected, planar, 2-colorable graph which is not acyclically 3-colorable.

While it is easy to obtain some bounds (quadratic in n) for the least number $a(n)$ of colors needed for acyclic colorings of all graphs with maximal valence n, it would be of interest to determine the exact values of $a(n)$. As mentioned above, $a(3) = 4$. It is not hard to show that $a(4) \leqq 6$, but this is probably not a best possible result. Indeed, we may make the following

CONJECTURE 2. $a(n) = n + 1$ *for all* $n \geqq 2$.

If true, this conjecture would imply the easily established observation of Motzkin [20] that the maximal possible "point arboricity" of graphs of maximal valence n does not exceed $[n/2] + 1$.

(8) Let $\mathscr{G}(m)$ denote the class of all graphs that possess no circuits of length m or less (in other words, have girth at least $m + 1$). Scott Niven raised the question what can be said about (m)-acyclic colorings of the members of $\mathscr{G}(m)$, where a k-coloring of a graph G is called (m)-acyclic if for every choice of m of the colors, the subgraph of G spanned by vertices of those colors is acyclic. While bounds $a_m(n)$ analogous to the $a(n)$ discussed above clearly exist for the number of colors needed for (m)-acyclic coloring of all members of $\mathscr{G}(m)$ having maximal valence n, no reasonable estimate for $a_m(n)$ is known. Without restrictions on the valence no bounds are possible, since it is a well known result of Erdös [5] that each $\mathscr{G}(m)$ contains graphs of arbitrarily large chromatic number. (See [10] for references to the literature on this topic and some related results.)

(9) Let $\mathscr{P}(m)$ denote the family of all planar graphs in $\mathscr{G}(m)$. The theorem of Grötzsch [7] (for more accessible proofs see [8], [21], and [22]) asserts that all graphs in $\mathscr{P}(3)$ are 3-colorable, while (as mentioned above) the graph of the cube shows that not all graphs in $\mathscr{P}(3)$ are acyclically 3-colorable.

CONJECTURE 3. *All planar graphs without* 3-*circuits are acyclically* 4-*colorable.*

It would also be of interest to determine whether all graphs in $\mathscr{P}(4)$ are acyclically 3-colorable.

(10) As shown by the complete graphs (which are the graphs of cyclic or neighborly 4-polytopes), there exists no analogue of Theorem 1 for 4-polytopal graphs since even their chromatic numbers have no finite bounds. However, denoting by $k(d)$ the least upper bound of the chromatic numbers of d-polytopal graphs in $\mathscr{G}(3)$, we venture:

CONJECTURE 4. $k(d) < \infty$ for all d.

CONJECTURE 5. $\sup \{k(d) \mid d \geqq 3\} = \infty$.

If Conjecture 4 is true for $d = 4$, it would make sense to inquire about the existence of a k such that each 4-polytopal graph in $\mathscr{G}(3)$ is acyclically k-colorable. Similar questions may obviously be asked for other values of d.

Problems analogous to the above may also be raised about d-polytopal graphs in $\mathscr{G}(4)$.

(11) Theorem 1 may probably be modified to hold for all graphs imbeddable into a given 2-manifold, but the details have not been worked out. Kronk [14] established, in analogy to Proposition 1, that every graph G imbeddable n a closed orientable 2-manifold of genus $g > 0$ has a $(2,2,\cdots,2)$-coloring, where the number of 2's is $[(\frac{1}{4}(9 + \sqrt{1 + 48g})]$. (For generalizations see [18] and [19].) The analogues of our Theorem 2 for graphs imbeddable in 2-manifolds of higher genus remain to be explored.

Also open are the questions concerning bounds for the chromatic number of graphs in $\mathscr{G}(m)$ that are imbeddable in various 2-manifolds, and of the possibilities of their acyclic coloring. The only results known in this direction (see [15]) deal with the extension of Grötzsch's theorem to graphs imbeddable in the torus.

(12) The idea of looking for k-colorings of graphs such that the subgraphs spanned by any two (or any r, or certain pairs of) colors have some desired properties can be exploited in countless directions. We shall not belabor the obvious, but would like to point to three open problems.

(i) What is the smallest k such that each planar graph has a k-coloring in which each bicolored path involves at most three vertices?

The existence of such a k may be easily deduced from Theorem 1, together with the estimate $k \leqq 2304 = 9 \cdot 2^8$.

In the formulation of the last two questions we shall say that a graph G imbedded in the plane is *accessible* provided some face of the imbedding is incident with all the vertices of G. (Such graphs are sometimes called "outerplane".)

(ii) What is the least integer k such that every planar graph G allows a k-coloring in which the subgraph spanned by any pair of colors is accessible?

Our Theorem 1 establishes that $k \leqq 9$. On the other hand, it is not hard to find examples that show $k \geqq 5$. We conjecture $k = 5$.

(iii) Does there exist a k such that every planar graph G is acyclically k-colorable and the vertices of each triplet of colors span an accessible subgraph of G?

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